2009 IEEE International Conference on Control and Automation
Christchurch, New Zealand, December 9-11, 2009

Variance Decomposition of Nonlinear Systems

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Abstract— **Until recently, the motivation for assessing closed-loop performance was to evaluate the effect of the controller on the output variation of linear systems. Far less has been written on assessing the multivariate noise case for nonlinear dynamic systems. In this paper, we develop a strategy to quantify the effects of different types of noise on nonlinear dynamic systems using an ANOVA-like decomposition method. We test the proposed strategy using nonlinear autoregressive moving average models which represent an important class of nonlinear systems. To reduce the computational burden, we use the Fourier Amplitude Sensitivity Test (FAST) method to estimate the partial variances which we can compare with the less efficient Monte-Carlo strategy. The results of this paper can be used in investment problems, biomathematics and control theory where multivariate disturbances are frequently encountered.**

I. INTRODUCTION

Analysis of variance (ANOVA) refers to the task of decomposing the variance of a response variable into contributions arising from the inputs, and assessing the magnitude and significance of each of their contributions. It has been successfully applied in control performance analysis, [1, 2] and extended for nonlinear systems in [3–5].

Traditionally the systems under consideration were static, but recently practitioners interested in variance decomposition have focused on dynamic systems such as the time series [6, 7], and univariate and multivariate linear dynamic systems. Analytical solutions for the linear univariate case were given by [8], while [9] extended this to include multivariate linear systems.

Not surprisingly far less has been written on assessing multivariate disturbance effects on the process performance for nonlinear systems due to the difficulty of adequately generalizing the complex structures of the nonlinear systems and subsequently solving them.

In this paper, we establish the contribution from stochastic input signals on the output variance of a nonlinear ARMA system. Knowing how the output varies with respect to variations of the disturbances yields insight into the behavior of the model and can assist the closed-loop performance assessment or variance reduction. For example, if a process has an additional measurement of some component of the disturbances, control engineers could use this extra information to improve the process performance such as implementation of feed-forward control or disturbance reduction. But before doing that, they must conduct an analysis of variance or variance decomposition.

For the purposes of this paper, we consider two types of disturbances: *dynamic disturbances*, ξ, which act directly on the dynamics, and *measurement disturbances*, ε, which are only added to the dynamics.

For linear systems the decomposition of the variance is achieved by the Impulse Response Function (IRF). However this strategy is not in general applicable for nonlinear systems, motivating [10] to derive an ANOVA-like decomposition for nonlinear static systems which is further developed in this paper to take into account the time dependence and initial conditions (ICs) of the underlying nonlinearities.

For the output of a static system represented as an analytic function of input variables, e.g., $Y = f(X_1, X_2, \cdots, X_p)$, the relative importance of the independent inputs can be quantified by the fractional variance which is defined as the fractional contribution to the output variance due to the uncertainties in inputs. This can be calculated using an ANOVA-like decomposition formula for the total output variance $\text{Var}(Y)$ [11, 12]:

$$V = \text{Var}(Y) = \sum_i V_i + \sum_i \sum_{j>i} V_{ij} + \cdots + V_{12\cdots p} \quad (1)$$

where $V_i = \text{Var}(E(Y|X_i = x_i))$ and

$$\begin{aligned} V_{ij} =& \text{Var}\left(E(Y|X_i = x_i, X_j = x_j)\right) \\ &- \text{Var}(E(Y|X_i = x_i)) - \text{Var}(E(Y|X_j = x_j)) \end{aligned} \quad (2)$$

and so on, where $E(Y|X_i = x_i)$ denotes the expectation of Y conditioned on X_i having a fixed value x_i, and V stands for variance over all the possible values of x_i.

The layout of this paper is as follows. In Section II, we consider a general nonlinear input-output model with multivariate disturbances and introduce a strategy for the variance decomposition. Two modifications of the ANOVA-like decomposition method are addressed. In Section III, two simulation examples are used to illustrate the essential features of the proposed methods. The paper concludes with a description of outstanding issues and limitations of the proposed methodology.

This work was supported by the Industrial Informal and Control Centre, The University of Auckland, New Zealand and the Department of Chemical Engineering, Queen's University, Canada.

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II. Variance Decomposition of Nonlinear ARMA Models

In this paper we are interested in nonlinear systems that are affected by both dynamic and measurement disturbances. Given the richness of behaviour that nonlinear systems can exhibit, for pragmatic reasons we restrict our attention to nonlinear input/output models such as the nonlinear autoregressive moving average with exogenous input shown in Fig. 1, (NARMAX) [13], described as

$$Y_t = f(Y_{t-1}, U_{t-b}, \xi_{1,t}, \cdots, \xi_{n_\xi,t}, \varepsilon_{1,t}, \cdots, \varepsilon_{n_a,t}) \qquad (3)$$

where the dynamic disturbance terms ξ, and the measurement disturbance terms ε, are assumed to be identically independent distributed (iid) variables with mean zero and variance $\sigma^2_{\xi_i}$ and $\sigma^2_{\varepsilon_i}$ respectively.

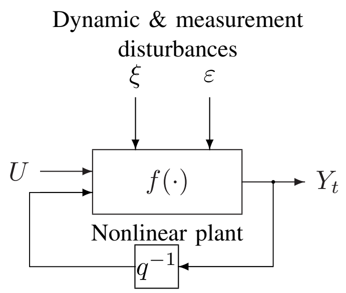

Fig. 1. The nonlinear model with dynamic and static measurement disturbances

A specific class of nonlinear model where there are stochastic coefficients is known as a random coefficient autoregressive model of order k, RCA(k) [13]. Such a model is used as a test case subsequently in section III-A.

A. Variance Decomposition

Although the nonlinear stochastic systems represented in Eq. (3) have been studied in some depth in the control and identification theory literature (e.g., [14–17]), the statistical analysis of these models is still in its infancy.

For linear systems, the effect of a disturbance can always be represented as an additive output disturbance regardless of where it actually appears in the system. For nonlinear systems however where superposition does not hold, quantifying the amount of disturbance becomes system dependent. However in the special case where the disturbances are measured, [18] quantifies the extent of the disturbance in the case of known dynamics, and [19] for the case of unknown dynamics.

For discussion simplicity, no driving-force in the dynamic system (which is often referred to the stochastic control problem) is considered initially in this paper. The extension to the more general situation is straightforward and will be illustrated by a simulation of a Volterra series model with two additive linear disturbances in section III-B.

An NARMA process with p different sources of disturbances $a_{i,t}$ for the single output case is

$$Y_t = f(\mathbf{Y}_{t-1}, \mathbf{a}_{1,t}, \cdots, \mathbf{a}_{p,t}) \qquad (4)$$

where the vectors $\mathbf{Y}_{t-1} \stackrel{\text{def}}{=} [Y_{t-1}, \cdots, Y_{t-n_y}]$ and $\mathbf{a}_{i,t} \stackrel{\text{def}}{=} [a_{i,t}, \cdots, a_{i,t-n_{a_i}}]$ collect old values. (Here $a_{i,t}$ are used to present both dynamic and measurement disturbances).

To decompose the variance of the nonlinear process using the proposed strategy, we need the following assumptions:

- Eq. (4) can be solved numerically subject to the initial conditions, I_0, to give Y_t for any choice of $a_{1,t}, \cdots, a_{p,t}$.
- the different disturbances are uncorrelated.
- for each source of disturbance $a_{i,t} \quad i = 1, 2, \cdots, p$, $a_{i,t} \quad t = 1, 2, \cdots$ are independent identically distributed (iid).
- the disturbances entering the systems after time $t = 0$ and the initial conditions I_0 are independent.
- the initial conditions vector I_0 is a random vector with probability density function $P(I_0)$.

We are interested in determining the sensitivity of the outputs Y_t, at each time interval in Eq. (4) to variations of each disturbance group $A_{i,t} = [a_{i,1}, \cdots, a_{i,t}], i = 1, \cdots, p$, noting also that the behaviour of nonlinear systems may depend strongly on the initial conditions. For this situation, we cannot use the ANOVA-like decomposition in Eqs. (1) and (2) directly since the initial conditions must be considered within the variance decomposition. Using the well-known variance decomposition theorem [20], we can decompose the variance of Y_t, $t = 1, 2, ..., n$ as:

$$\text{Var}[Y_t] = E_{I_0}[\text{Var}_{A_t}[Y_t|I_0]] + \text{Var}_{I_0}[E_{A_t}[Y_t|I_0]] \qquad (5)$$

where $A_t = [A_{1,t}, A_{2,t}, ..., A_{p,t}]$ denotes all of disturbances entering the system from time 1 to time t. $E_{I_0}[\cdot]$ and $\text{Var}_{I_0}[\cdot]$ denotes the expectation and variance of $[\cdot]$ with respect to I_0 respectively. The second term on the right-hand side of Eq. (5) is the fractional contribution to the output due *only* to the uncertainties of the initial conditions. The first term in the right-hand side of Eq. (5) is the variance contribution to the output due to the disturbances $A_t = [A_{1,t}, A_{2,t}, ..., A_{p,t}]$ with initial conditions uncertainty. From Eq. (5), it is straightforward to obtain $\text{Var}[Y_t] \geq E_{I_0}[\text{Var}_{A_t}[Y_t|I_0]]$. The special situation, $\text{Var}[Y_t] = E_{I_0}[\text{Var}_{A_t}[Y_t|I_0]]$ will be discussed in the following paragraphs.

The conditional variance given initial conditions I_0, $\text{Var}_{A_t}[Y_t|I_0]$, can be decomposed directly using the ANOVA-like decomposition method as:

$$V_{A_t}(Y_t|I_0) = \sum_i V_{A_{i,t}}|I_0 + \sum_i \sum_{j>i} V_{A_{i,t}A_{j,t}}|I_0 + \cdots + V_{A_{1,t}\cdots A_{p,t}}|I_0 \qquad (6)$$

where

$$\begin{aligned} V_{A_{i,t}}|I_0 &= V_{A_{i,t}}(E_{(A_{i,t})'}(Y_t|A_{i,t}, I_0)) \\ V_{A_{i,t}A_{j,t}}|I_0 &= V_{A_{i,t}A_{j,t}}(E_{(A_{i,t}A_{j,t})'} \\ &\quad (Y_t|A_{i,t}, A_{j,t}, I_0)) - V_{A_{i,t}}|I_0 - V_{A_{j,t}}|I_0 \\ &\vdots \end{aligned} \qquad (7)$$

where the index $(A_{i,t})'$ stands for "all $A_t = [A_{1,t}, \cdots, A_{p,t}]$ but $(A_{i,t})$" – that is, complementary to $(A_{i,t})$.

The variance decomposition with consideration of the initial conditions can be obtained by simply calculating the expectation of the conditional variance decomposition in Eq. (6) with respect to the initial conditions I_0. This procedure is not necessary if the initial conditions have, or can be approximately assumed to have, a linear relationship with the output Y_t . The variance decomposition can be calculated with the results of the conditional variance decomposition in Eq.(6) based on the mean values of initial conditions. Further information about this topic can be found in [21]. Since the ANOVA-like decomposition method is model independent, using this method has the advantage that the the nonlinear system model need not be known.

Special case:

If the stochastic process in Eq.(4) is a geometrically ergodic Markov chain, the variance decomposition in Eqs. (4) can be succinctly obtained by applying the ANOVA-like decomposition method. The criteria guaranteeing a Markov chain to be geometrically ergodic can be found in [13, P. 127]. If the stochastic process in Eq. (4) is a geometrically ergodic Markov chain, then given the initial conditions $I_0 = [y_0^*, a_{1,0}^*, \cdots, a_{p,0}^*]$ there exists a limiting probability as

$$\lim_{t\to\infty} P(Y_t|I_0) = \pi \geq 0 \tag{8}$$

The limiting probability π is independent of the initial conditions.

Since for a geometrically ergodic Markov chain, $E_{A_t}[Y_t|I_0]$ is equal to a constant value for any initial conditions for $t \to \infty$, the term $Var_{I_0}[E_{A_t}[Y_t|I_0]]$ in Eqn. 5 is zero. The limiting variance of output Y_t now simplifies to

$$\text{Var}[Y_{t\to\infty}] = E_{I_0}[\text{Var}_{A_t}[Y_{t\to\infty}|I_0]] \tag{9}$$

Since the output $Y_{t\to\infty}$ is independent of the initial conditions I_0, Eq. (9) can be written as:

$$\text{Var}[Y_{t\to\infty}] = \text{Var}_{A_t}[Y_{t\to\infty}|I_0] \tag{10}$$

The results of variance decomposition using the method in Eq. (6) will not depend on the initial conditions. A finite series of m memory terms, is used for the variance decomposition. Since the polynomial NARMA models are most often used to represent nonlinear systems, auto-correlation, cross-correlation and cross bi-correlation between output and disturbances can be used to find a suitable m.

B. Estimation Methods for $Var_{A_t}[Y_t|I_0]$

Monte-Carlo (MC) techniques are one way to bypass the intractability of analytically computing the variance decomposition for Eq. (6) for general nonlinear time series, [22, 23].

Efficient numerical methods are required for large scale problems [24]. The FAST [25, 26] and Sobol's [27] methods have been developed to cope with this dimensionality problem. Further details on the FAST methods used in this approach can be found in [10].

III. Simulation experiments

This section presents two simulation experiments to show the effectiveness of the proposed strategy. The first demonstrates the variance decomposition for a system with random coefficients, while the second example illustrates the case where two linear stochastic terms are added to a nonlinear Volterra model.

A. A Random Coefficient Autoregressive Model

Consider the first order random coefficient autoregressive RCA(1) model:

$$Y_t = (\alpha + a_{1,t})Y_{t-1} + a_{2,t} \tag{11}$$

where $a_{1,t}, a_{2,t}$ are i.i.d. normally distributed with mean zero and variance σ_1^2, σ_2^2, independent of initial conditions Y_0 and α is a real constant. Using the criteria given in [13], $\alpha^2 + \sigma_1^2 < 1$ is a sufficient condition for model (11) to be ergodic.

The infinite solution of Y_t is given by:

$$Y_t = \sum_{j=0}^{\infty} \pi_j a_{2,t-j} \tag{12}$$

where $\pi_0 = 1$, and

$$\pi_j = \prod_{i=0}^{j-1} (\alpha + a_{1,t-i}), \quad j = 1, 2, \cdots \tag{13}$$

Letting $\tau = \alpha^2 + \sigma_1^2 < 1$, it is straightforward to show:

$$E[Y_{t\to\infty}] = 0 \quad \text{and} \quad \text{Var}(Y_{t\to\infty}) = \frac{\sigma_2^2}{1-\tau^2} \tag{14}$$

Using the method in Eqs. (1) and (2), the analytical solution for the variance decomposition at time $t \to \infty$ is:

$$V_1 = 0, \quad V_2 = \frac{\sigma_2^2}{1-\alpha^2}, \quad V_{12} = \frac{\sigma_1^2\sigma_2^2}{(1-\alpha^2)(1-\tau^2)} \tag{15}$$

Using the model Eqn. 11 with parameter $\alpha^2 = 0.6$, and Eqn. 15 with parameters , $\sigma_1^2 = 0.3$ and $\sigma_2^2 = 0.4$ gives theoretical values for the variance decomposition $V_1 = 0, V_{12} = 0.625$ and $V_{12} = 0.2492$.

Since this example is initial condition independent according to the criteria [13], Y_0 is set to zero. Since the initial conditions include terms $Y_{t-1}, a_{1,t}Y_{t-1}$, and $a_{2,t}$, $t \leq 1$, the auto-correlation r_{yy}, cross-correlation r_{ya_2} and cross bi-correlation t_{yya_1} defined in Section II-A are estimated using 500 realizations. The results are shown in Fig. 2. It shows that this RCA(1) model's memory can be adequately approximated to be 10 effectively saying that the effects of the initial conditions $Y_{t-1}, a_{1,t}, Y_{t-1}$, and $a_{2,t}$, $t \leq 1$ on the present output value Y_{10} would be insignificant.

Now the infinite series described in Eq. (12) can be truncated to

$$Y_{t\to\infty} \simeq Y_{10} = \sum_{j=0}^{m=10} \pi_j a_{2,t-j} \tag{16}$$

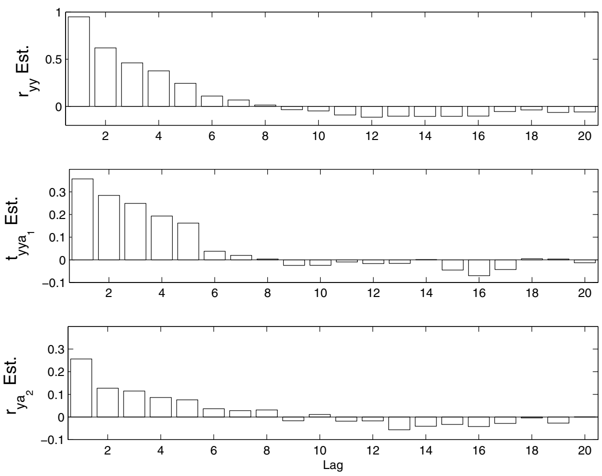

Fig. 2. Auto-correlation, cross-correlation and cross bi-correlation for the RCA(1) model, Eqn. 11, under consideration. This leads us to choose 10 as a suitable memory length for the nonlinear model.

where $\pi_0 = 1$ and

$$\pi_j = \prod_{i=0}^{j-1} (\alpha + a_{1,t-i}), \quad j = 1, 2, \cdots \tag{17}$$

The variance of Y_{10} is:

$$\text{Var}(Y_{10}) = \frac{(1 - \tau^{20})\sigma_2^2}{1 - \tau^2} \tag{18}$$

For our simulation, Var$(Y_{10}) = 1.148$ is accounting for 97.6% of the limiting variance Var$(Y_{t\to\infty}) = 1.176$.

To decompose the variance for Y_{10}, we use the different choices of frequencies shown in Table I. N_s denotes the sample size used in FAST. Fig. 3 shows an example of the extended FAST method applied to the RCA(1) model with finite memory 10. One hundred estimates, obtained using different starting points, of the partial and total variances are computed. The boxplots of their summary statistics, for each disturbance group, are plotted against the sample size N_s. The estimates converge to the analytical values and the precision of the estimates increases, as the number of samples (or the spread of frequencies) increases.

TABLE I

SETS OF FREQUENCIES OBTAINED BY USING THE AUTOMATED ALGORITHM

Sim. No.	N_s	High Freq.	Complementary Freq. Max	Low Freq.	Step
1	65	8	1	{1,1,1,1,1,1,1,1,1,1}	0
2	641	80	10	{1,2,3,4,5,6,7,8,9,10}	1
3	1217	152	19	{1,3,5,7,9,11,13,15,17,19}	2
4	2369	296	37	{1,5,9,13,17,21,25,29,33,37}	4

The results of variance decompositions for $Y_t, t = 1, \cdots, 15$ are plotted in Fig. 4. It shows that the estimates of the variance decomposition results appears to converge to the true values when the time horizon increases.

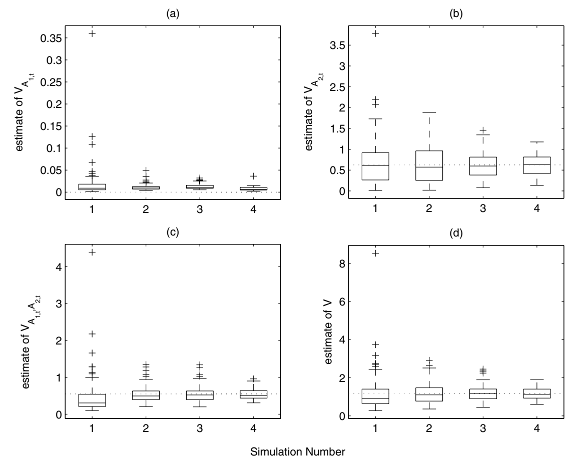

Fig. 3. Boxplots of 100 estimates of the variance decomposition for the RCA(1) model; analytical values of the partial and total variances are shown by dotted lines.

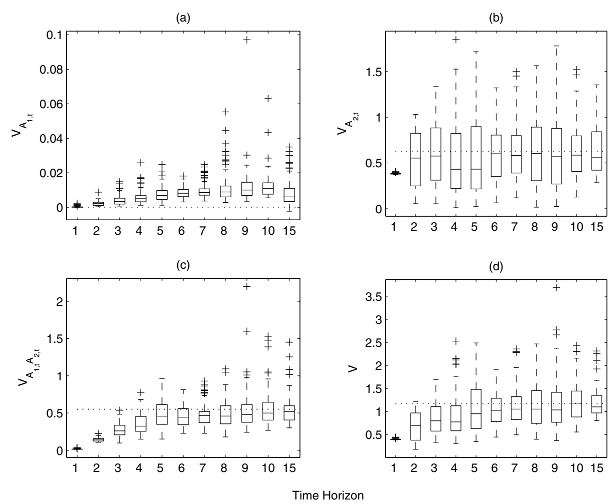

Fig. 4. Boxplots of 100 estimates of the variance decomposition of Y_t, $t = 1, \cdots, 15$ for the RCA(1) model ; analytical values of the partial and total variances $t \to \infty$ are shown by dotted lines.

B. A Volterra Model

This example illustrates the variance decomposition for a nonlinear Volterra model subjected to two additive linear disturbances as shown in Fig. 5.

The model can be expressed as:

$$\begin{aligned} Y_t = &0.2U_{t-3} + 0.3U_{t-4} + U_{t-5} + 0.8U_{t-3}^2 + 0.8U_{t-3}U_{t-4} \\ &- 0.7U_{t-4}^2 - 0.5U_{t-5}^2 - 0.5U_{t-3}U_{t-5} \\ &+ D_{1,t} + D_{2,t} \end{aligned} \tag{19}$$

where the disturbance $D_{1,t}$ is the measured disturbance which is in the form of an ARMA(2,0,0) process:

$$D_{1,t} = \frac{a_{1,t}}{1 - 1.6q^{-1} + 0.8q^{-2}} \tag{20}$$

and the second disturbance $D_{2,t}$ is the unmeasured distur-

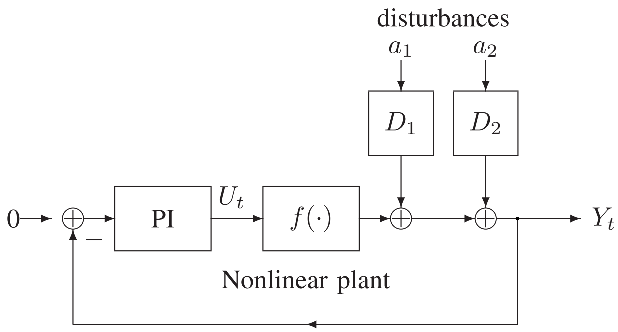

Fig. 5. The nonlinear closed loop system with two sources of disturbances

bance which can be represented as an AR(1) process:

$$D_{2,t} = \frac{a_{2,t}}{1 - 0.9q^{-1}} \tag{21}$$

$a_{1,t}$ and $a_{2,t}$ are the i.i.d. normal variables with zero mean and variance 0.03 and 0.05 respectively. The variances of the disturbances $D_{1,t}$ and $D_{2,t}$ are consequently equal to 0.1997 and 0.2632 respectively, and the disturbances $D_{1,t}$ and $D_{2,t}$ are uncorrelated. A PI controller $U_t = -\frac{0.3-0.2q^{-1}}{1-q^{-1}}(Y_t)$ is used to close the loop. A closed-loop data set consisting of five hundred samples for $D_{1,t}$, $D_{2,t}$, U_t and Y_t is shown in Fig. 6.

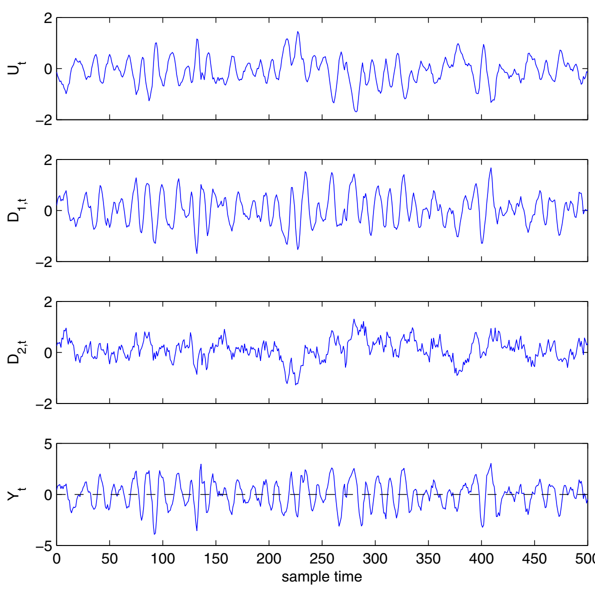

Fig. 6. 500 samples of the closed-loop Volterra system subjected to measured and unmeasured disturbances

1) Variance Decomposition with Initial Conditions $I_0 = 0$: Since it is impossible to obtain an analytical solution for the variance decomposition, a Monte Carlo (MC) method is used to estimate the partial / total variances. Based on the initial conditions $I_0 = 0$, the variation of output Y_t, $t = 1, 2, \ldots, 40$ is shown Fig. 7. From Fig. 7, we can observe that the variations of output Y_t between time $t = 20$ and time $t = 40$ are not significantly different leading us to choose an appropriate memory length of 20.

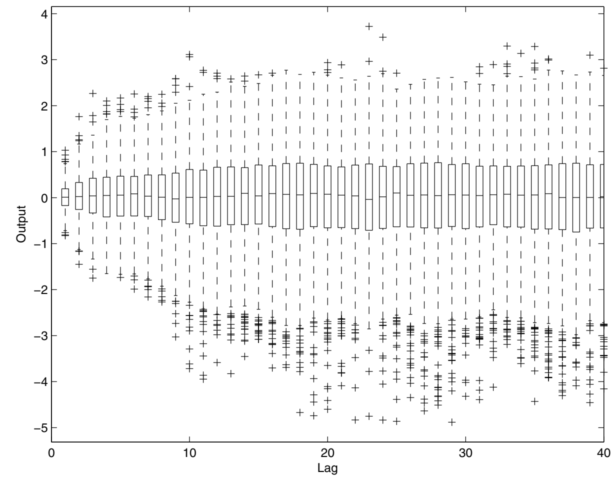

Fig. 7. The box plots of output Y_t for the first 40 samples. This indicates 20 is a suitable memory length for the model expressed by Eqn. 19.

The procedures for estimating the partial variance V_1 is shown in the following steps: i) one sample set of $A^1_{1,t} = [a_{1,1}, ..., a_{1,20}]$ are generated, ii) the other sample set of $A^1_{2,t} = [a_{2,1}, ..., a_{2,20}]$ are generated, the output y^1_{20} is calculated, iii) the step ii) is repeated for two hundred times to collect the output $y^i_{20}, i = 1, ..200$. iv) estimate the condition mean $E(y_t|A^1_{1,t})$ using the output data from step iii), v) repeat steps i)-iv) two hundred times and estimate the partial variance V_1 using two hundred means. The same procedures must be repeated for estimating the partial variance V_2. The estimates of the partial and total variances are listed in column 2 (The values in parentheses are the standard deviations of the estimates) of Table II.

TABLE II

ESTIMATES OF PARTIAL/TOTAL VARIANCE USING MC AND FAST METHODS WITH CONSTANT INITIAL CONDITIONS

	V_1	V_2	V_{12}	V
MC	0.85	0.21	0.03	1.08
FAST	0.84 (0.71)	0.19 (0.17)	0.03 (0.10)	1.06 (0.76)

The partial and total variances are also estimated using the FAST method with a transformation function for a normal variable. The high frequency of 16 is assigned to the unmeasured disturbance $D_{2,t}$ and a low frequency of 2 to the measured disturbance $D_{1,t}$. The sample size is 129 for each estimation. The estimation is repeated two hundred times. Unlike the MC methods where he estimations must be done individually, the partial and total variances can be estimated simultaneously from the FAST method for the two factor case. The estimates of the partial and total variances using the FAST method are also listed in column 2 of Table II (The values in parentheses are the standard deviations of the estimates).

The fact that the calculation load for the FAST scheme was approximately 25 times less has a large impact on the following simulation which includes the consideration of initial condition uncertainty. The comparative box plots of the quality estimates are shown in Fig. 8.

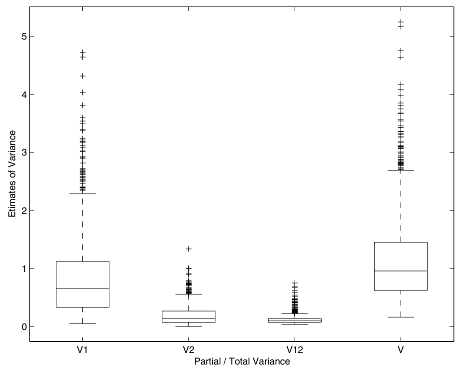

Fig. 8. The comparative box plots of the quality estimates of partial/total variance using the FAST method

A sample of size $(200 \times 200 \times 20)$ is used to estimate the V_1 using the MC method. The first 200 samples for $A_{2,20}$ are used to estimate $E_{A_{2,20}}[Y_{20}|A_{1,20}]$ and the second 200 samples are used to estimate $V_{A_{1,20}}(E_{A_{2,20}}[Y_{20}|A_{1,20}])$. The third sample size 20 is from memory length 20. The estimates of the partial and total variances using the MC method are also listed in column 2 in Table II. The estimate of V_1 using the FAST method with only $(200 \times 129 \times 20)$ sample size is close to the MC estimate. The sample size for FAST method means: 129 samples for FAST method, 20 is the memory length, and 200 is the repeat number for the FAST method. Furthermore, with the same sample set, the FAST method can compute all values of V_1, V_2, V_{12} and V at the same time. The new sample set is required by the MC method for each new estimate of V_2, V_{12} and V. The MC method is computationally more expensive in terms of model evaluations.

2) Variance Decomposition with Uncertain Initial Conditions I_0: For this simulation, the burn-in period is 30. The sample size $(200 \times 200 \times 200 \times 20)$ is used to estimate the V_1 using the MC method. The first sample size 200 is for the estimation of the partial variance V_1 with consideration of initial condition uncertainty $E_{I_0}[V_{A_{1,20}}(E_{A_{2,20}}[Y_{20}|A_{1,20}])]$. The rest sample sizes have the same definition used in the above simulation. Similarly, the sample size $(200 \times 200 \times 129 \times 20)$ is used for the FAST method. The effects of the uncertainties of the initial conditions on the output variance decomposition are listed in Table III. While the FAST strategy is around 16 times faster in this example, this comes at a slight increase in the uncertainty of the variance estimates. The comparative box plots of the quality estimates using the FAST method are shown in Fig. 9.

Since the expected values of the initial conditions are equal to zero, the results of variance decomposition with constant initial conditions listed in Table II can be considered as the variance decomposition of the expected values of the initial conditions for example

TABLE III

ESTIMATES OF PARTIAL / TOTAL VARIANCE USING MC AND FAST METHODS WITH UNCERTAIN INITIAL CONDITIONS

	V_1	V_2	V_{12}	V
MC	0.73	0.21	0.07	1.01
FAST	0.73 (0.64)	0.21 (0.18)	0.08 (0.08)	1.01 (0.71)

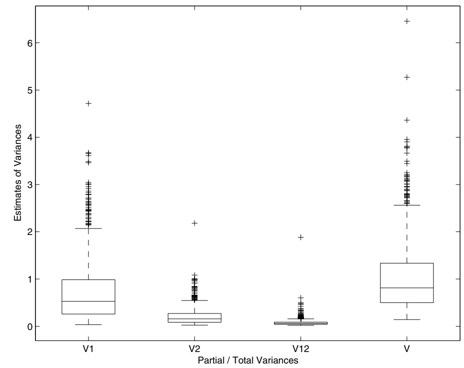

Fig. 9. The comparative box plots of the quality estimates of partial/total variance with the uncertain initial conditions using the FAST method.

$V_{A_{1,20}}(E_{A_{2,20}}[Y_{20}|(A_{1,20}, E[I_0])]$. The results in Table III are the expected values of the initial conditions of the variance decomposition $E_{I_0}[V_{A_{1,20}}(E_{A_{2,20}}[Y_{20}|A_{1,20}])]$. In general these two calculations are not equal. The results in Table II and Table III illustrate these inequalities. The differences between these two terms may or may not be significant, it will depend on the model structure and disturbance statistics. The differences between these two terms with different time horizons are listed in Table IV. The data in column 3 in Table IV are obtained using the MC method and the data in column 4 are calculated using the FAST method with the same parameter values such as frequency used in the previous simulation.

TABLE IV

ESTIMATES OF PARTIAL / TOTAL VARIANCE OF THE CONSTANT INITIAL CONDITIONS AND THE UNCERTAIN INITIAL CONDITIONS FOR THE DIFFERENT TIME HORIZONS

Horizon	*Variance*	*Constant IC*	*Uncertain IC*
$t = 10$	V	0.82	0.47
	V_1	0.61	0.29
	V_2	0.18	0.16
	V_{12}	0.03	0.02
$t = 20$	V	1.09	1.01
	V_1	0.85	0.73
	V_2	0.21	0.21
	V_{12}	0.03	0.08
$t = 30$	V	1.15	1.17
	V_1	0.85	0.86
	V_2	0.19	0.19
	V_{12}	0.11	0.12

From Table IV, we can see that the differences between these two terms will become smaller as the time horizon increases for this simulation example. An interesting phenomena in Table IV is that the conditional output variance given constant initial conditions is much less than the output variance for the output Y_{10} case. It means that the results of the variance decomposition based on uncertain initial conditions may be significantly different from the results based on constant initial conditions.

IV. CONCLUSIONS AND FUTURE WORK

A. Conclusions

This paper has provided a preliminary analysis of variance decomposition for MISO nonlinear processes. We have addressed the case where there is no cross-correlation between the disturbances within the nonlinear systems which can be represented by NARMAX models. We have shown that the variance decomposition of the nonlinear time series and nonlinear stochastic systems can be estimated using the ANOVA-like decomposition in Eq. (1). Since for nonlinear stochastic/dynamic systems the variance decomposition is now dependent on the initial conditions, a modified ANOVA-like decomposition method is proposed to cope with the initial condition uncertainty. Applications of the methodology to the examples indicate that this approach gives very credible estimates of the variance decomposition.

B. Future Work

The variance decomposition for the nonlinear dynamic/stochastic systems and time series discussed in this paper is based on the assumption that the different disturbances are uncorrelated. This assumption is not always applicable in practice. The investigation into the effects of cross-correlated disturbances on analysis of variance for nonlinear MISO systems may be necessary for the extension of nonlinear variance decomposition problems.

V. ACKNOWLEDGMENTS

Financial support to this project from the Industrial Information and Control Centre, Faculty of Engineering, The University of Auckland, New Zealand is gratefully acknowledged.

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